NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2543

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INVESTIGATION OF TURBULENT BOUNDARY LAYER OVER A FLAT PLATE IN COMPRESSIBLE FLOW BY THE SIMILARITY THEORY

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INVESTIGATION OF TURBULENT BOUNDARY LAYER OVER A FLAT PLATE IN COMPRESSIBLE FLOW BY THE SIMILARITY THEORY

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SUMMARY

Investigation of the turbulent-boundary-layer flow over a flat plate in compressible flow is carried out on the basis of the scheme established in NACA TN 2542. By averaging the Navier-Stokes equations, differential equations for the mean flow are obtained. A temperature-velocity relation follows without a specified form of the length scale. To derive the velocity distribution in the boundary layer, a choice of the length scale has to be made. The temperature-velocity relation reduces to Reynolds' analogy and the velocity distribution goes back to Von Kármán's logarithmic law for the special case of incompressible flow.

There are essentially three universal constants, arising out of the correlations in the energy equation, to be determined by comparing with suitable experiments of the temperature-velocity relation at any known Mach number and heat transfer at wall. The behavior at other Mach numbers and heat-transfer conditions may then be readily predicted. Because of the lack of accurate experimental data, attempts to carry out such determinations are not included in the present report.

INTRODUCTION

As part of an investigation of Von Kármán's similarity theory and its extension to compressible flows, the theory for incompressible flows was examined critically in reference 1 by using modern concepts. It was found that the original form of the theory is supported by modern concepts. In reference 2, the theory was extended to the case of compressible flows and it was found that the analysis could be carried through with additional approximations but without any modification of the basic concepts. In the present paper the problem of compressible flow in a turbulent boundary layer is considered on the basis of the scheme established in reference 2. It is shown how the influence of the Mach number (e.g., on the velocity and temperature distribution) can be predicted from the theory after the constant coefficients in the theory are determined by one set of experimental measurements.

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VELOCITY-TEMPERATURE RELATION

Attempts will now be made to deduce relationships among the mean quantities and their distributions within the boundary layer. To do so one returns back to the complete equations of motion (equations (26) to (29) of reference 2) instead of working with the "localized" equations where the observer rides with the mean velocity. These equations are averaged so that only mean quantities and the correlations of the fluctuating quantities appear.

A strictly parallel flow will be considered. The justification of its application to the case of a boundary layer will be discussed later. By using the continuity equation, it is easy to show that

$$\overline{\left(\rho \frac{D}{Dt} F\right)} = \frac{\partial}{\partial y} \overline{\rho v F} \tag{1}$$

when the mean flow depends on y only. Here the bar denotes the average of the quantity under it and F stands for any function. (See appendix A for definitions of symbols.) Thus, equation (26) of reference 2 leads to

$$\left.\begin{aligned} &\frac{\partial}{\partial y} \overline{\rho u v} = 0 \\ &\frac{\partial}{\partial y} \overline{\rho v^2} = \frac{\partial \overline{p}}{\partial y} \\ &\frac{\partial}{\partial y} \overline{\rho v w} = 0 \end{aligned}\right\} \tag{2}$$

There follows immediately, by integration,

$$
\left.
\begin{aligned}
\overline{\rho u v} &= \text{Constant} = \tau_o \\
\overline{\rho v^2} &= \overline{p} - \overline{p_1} \\
\overline{\rho v w} &= \text{Constant} = \tau_{oz}
\end{aligned}
\right\} \tag{3}
$$

where τ_o is the shearing stress at the plate in the x-direction, τ_{oz}, the shearing stress at the plate in the z-direction, and $\overline{p_1}$, the pressure outside the boundary layer. One expands the averaged quantities in equations (3) and finds that triple correlations are involved as well as the mean-flow variables and double correlations. In usual cases the triple correlations are taken to be of higher order than the double correlations. For example,

$$
\begin{aligned}
\overline{\rho' u' v'} &\approx O\left(\sqrt{\overline{(\rho')^2}}\ \overline{u'v'}\right) \\
&\approx \sqrt{\overline{(\rho')^2}}\, O(\overline{u'v'}), \ll \overline{\rho}\ \overline{u'v'}
\end{aligned} \tag{4}
$$

A word of caution, however, may be in order here to warn against the overconfidence in such estimations. It is nothing more than a plausible guess of the individual terms, and sometimes a combination of the individual terms may very well invalidate the result. The discussion in reference 2 of the dilatation e serves as a good example.

Nevertheless, it is perhaps admissible in the present case to drop triple correlations involving ρ' on the basis of the estimations like equation (4). If one concedes that the effect of the triple correlations is not negligible but relatively small, then the validity of the theory, being an approximate one, might not be seriously impaired because of the omission. By leaving out the triple correlations, equations (3) are expanded into the following:

$$\overline{\rho}\ \overline{u'v'} + \overline{u}\ \overline{\rho'v'} = \tau_o \tag{5}$$

$$\begin{aligned}\overline{p} &= \overline{p_1} + \overline{\rho}\ \overline{(v')^2} \\ &= \overline{p_1}\left(1 + \gamma M_1{}^2 \frac{\overline{\rho}}{\overline{\rho_1}} \frac{\overline{(v')^2}}{\overline{u}^2}\right)\end{aligned} \tag{6}$$

$$\overline{\rho}\ \overline{v'w'} = \tau_{oz} \tag{7}$$

where subscript 1 denotes the free-stream quantity. From the equation of continuity, one has

$$\frac{\partial}{\partial y} \overline{\rho v} = 0$$

hence

$$\overline{\rho'v'} = \text{Constant} = m \tag{8}$$

where m is evidently the mass transfer through the wall. Likewise from the equation of energy (equation (28) of reference 2)

$$c_p \frac{\partial}{\partial y} \overline{\rho v T} - R \frac{\partial \overline{p}}{\partial y} = \overline{\mu \epsilon}$$

which yields, by omitting the triple correlation $\overline{\rho' v' T'}$ and integrating between 0 and y,

$$c_p\left(\overline{\rho}\ \overline{v'T'} + \overline{T}\ \overline{\rho'v'}\right) - R\left(\overline{p} - \overline{p_o}\right) = \int_{\delta^*}^{y} \overline{\mu\epsilon}\, dy + q + q^* \tag{9}$$

where $\overline{p_o}$ is the pressure at the wall (cf. equation (6)), δ^*, the effective thickness of the laminar sublayer, q, the heat transfer at the wall, and $q^* = \int_0^{\delta^*} \overline{\mu\epsilon}\, dy$, the heat generated within the laminar sublayer (cf. appendix B for discussion of q^*).

One may now make the following observations: In equation (8), the turbulent mass transfer m vanishes if there is no addition or subtraction of mass through the plate. For a nonporous flat plate, then, equation (5) leads to

$$\bar{\rho}\, \overline{u'v'} = \tau_o \tag{10}$$

One may note that equation (10) is also the basic equation in references 3 and 4, where the role of $\overline{\rho'v'}$ was not mentioned. In reference 5, the term $\overline{\rho'v'}$ is kept and given no physical interpretation.

With the previous assumption of small turbulence level $\overline{(v')^2}\big/\overline{u^2} \ll 1$, when the free-stream Mach number is of order unity, equation (6) reduces to

$$\bar{p} \approx \overline{p_1} \tag{11}$$

as usually accepted for boundary-layer flows, and the term $\bar{p} - \overline{p_o}$ drops out in equation (9). It is seen, however, that equation (11) ceases to be true if

$$M_1^2 \times \frac{\overline{(v')^2}}{\overline{u^2}} \approx O(1)$$

Equation (7) is merely the statement of constancy of the transverse shear in the z-direction. It does not influence the mean motion in the xy-plane.

The next step is to introduce the similarity theory and represent the fluctuations in terms of the scales. One obtains from equation (10)

$$\bar{\rho} v_o^2 = \tau_o \tag{12}$$

by absorbing the correlation constant $\overline{u'v'}$ into v_o, the similarity scale for velocity fluctuations. Similarly from equation (9), with T' broken into T_1' and T_2' according to reference 2,

$$c_p\bar{\rho}\left(\overline{v'T_1'} + \overline{v'T_2'}\right) = \int_{\delta*}^{y} \alpha_1\bar{\rho}v_o^2 \frac{dy}{t_o} + q + q* \tag{13}$$

where α_1 is a correlation constant for the dissipation and use has been made of the relation

$$\overline{\left(\frac{\mu\epsilon}{\rho}\right)} = \overline{\mu\epsilon\left[1 + O(\rho'/\bar{\rho})\right]}\Big/\bar{\rho}$$

$$\approx \overline{\mu\epsilon}/\bar{\rho} \propto v_o^2/t_o$$

Introducing scales θ_1 and θ_2 for T_1' and T_2', respectively, and absorbing the correlations into proportional constants, α_2 and α_3

$$c_p\bar{\rho}\left(\overline{v'T_1'} + \overline{v'T_2'}\right) = \bar{\rho}v_o\left(\alpha_2 l_o \frac{d\bar{T}}{dy} + \alpha_3 v_o^2\right)$$

Using equation (12) and the definition of t_o in equations (32) of reference 2,

$$\int_{\delta*}^{y} \alpha_1\bar{\rho}v_o^2 \frac{dy}{t_o} = \alpha_1\tau_o \int_{\delta*}^{y} d\bar{u}$$

$$= \alpha_1\tau_o(\bar{u} - \overline{u*})$$

with $\overline{u*}$ representing the velocity at the edge of the laminar sublayer. Hence equation (13) becomes

$$\bar{\rho}v_o\left(\alpha_2 l_o \frac{d\bar{T}}{dy} + \alpha_3 v_o^2\right) = \alpha_1\tau_o(\bar{u} - \overline{u*}) + q + q*$$

or

$$\alpha_2 \frac{d\overline{T}}{d\overline{u}} + \alpha_3 v_o = \alpha_1(\overline{u} - \overline{u^*}) + \frac{q + q^*}{\tau_o} \tag{14}$$

where again the relations (12) and equations (32) of reference 2 have been used.

The quantity v_o may be put in terms of $\overline{T}$:

$$v_o = \sqrt{\frac{\tau_o}{\overline{\rho}}} = \beta\sqrt{\overline{T}} \tag{15}$$

with $\beta = \sqrt{R \frac{\tau_o}{\overline{p}}}$. Also, since $q^* = \tau_o\overline{u}^*$ (see appendix B), one may introduce an "effective" velocity

$$\overline{u_e} = \overline{u} - \left(1 - \frac{1}{\alpha_1}\right)\overline{u^*} \tag{16}$$

Then the final form of the differential equation between $\overline{T}$ and $\overline{u}$ is reduced from equation (14) into

$$\alpha_2 \frac{d\overline{T}}{d\overline{u}_e} + \alpha_3\beta\sqrt{\overline{T}} = \alpha_1\overline{u}_e + q \tag{17}$$

Equation (17) is derived without a specific assumption on the length scale l_o. The constant α_1 represents the heating due to dissipation, as a part of the external work $\tau_o\overline{u}$; α_2 represents both the mixing phenomenon and the compressibility; and α_3 reflects the combined effect of the compressibility and the dissipation. On a closer look at expression (16), it is revealed that the factor $\left(1 - \frac{1}{\alpha_1}\right)$ arising from the heating effect is originated from the idealized situation of an entirely turbulent outer boundary layer with similarity and an inner laminar viscous sublayer. In the sublayer all the energy supplied by the external work must go into heat through dissipation, hence the constant unity appears. In the turbulent layer, part of the energy is transferred or turned into turbulent energy, therefore only a portion α_1 is dissipated. If a transition region between the two idealized regions had been assumed, there would be no discontinuity in the picture but a term of the order $\left(1 - \alpha_1\right)\overline{u^*}$ would still be present in the heating because of dissipation.[1]

Equation (17) may be nondimensionalized by writing

$$\left.\begin{aligned} \theta &= \overline{T}\Big/\overline{T_1} \\ \widetilde{u} &= \overline{u_e}\Big/\overline{u_1} \\ c_f &= \tau_o\Big/\frac{1}{2}\,\overline{\rho_1}\,\overline{u_1}^2 \\ c_q &= q\Big/\overline{\rho_1}\,\overline{u_1}\left(\overline{T_o} - \overline{T_1}\right)c_v \end{aligned}\right\} \tag{18}$$

where subscript 1 again refers to the free-stream value and $\overline{T}_o$ is the temperature at the wall. Then there follows

$$\frac{d\theta}{d\widetilde{u}} = M_1^{\,2}\left(A_1\widetilde{u} + A_2\sqrt{\theta}\right) + Q \tag{19}$$

---

[1]To see this, one could assume a linear transition of the dissipation parameter such that

$$\begin{aligned} \overline{\mu\epsilon} &= \tau_o\,\frac{d\overline{u}}{dy}, \quad 0 \leqq y \leqq c_1\delta^*, \quad c_1 < 1 \\ &= \left[1 + \left(\alpha_1 - 1\right)\frac{\overline{u} - \overline{u_{c_1}}}{\overline{u_{c_2}} - \overline{u_{c_1}}}\right]\tau_o\,\frac{d\overline{u}}{dy}, \quad c_1\delta^* \leqq y \leqq c_2\delta^*, \quad c_2 > 1 \\ &= \alpha_1\tau_o\,\frac{d\overline{u}}{dy}, \quad c_2\delta^* \leqq y \end{aligned}$$

where $c_1\delta^* \leqq y \leqq c_2\delta^*$ is the transition region and $\overline{u_{c_1}}$ and $\overline{u_{c_2}}$ are the respective velocities at the ends of the transition. Then the term corresponding to $\left(1 - \frac{1}{\alpha_1}\right)\overline{u^*}$ is $\left(1 - \frac{1}{\alpha_1}\right)\left(\frac{\overline{u_{c_1}} + \overline{u_{c_2}}}{2}\right)$.

where

$$\left.\begin{aligned} A_1 &= \gamma R \frac{\alpha_1}{\alpha_2} \\ A_2 &= \gamma R \sqrt{c_f/2}\, \frac{\alpha_3}{\alpha_2} \\ Q &= 2\, \frac{c_v}{\alpha_2}\left(\frac{\overline{T_o}}{\overline{T_1}} - 1\right) c_q/c_f \end{aligned}\right\} \tag{20}$$

Equation (19) is the fundamental equation of the present theory for the variation of mean temperature with mean velocity. In the spirit of the local similarity, since the turbulence pattern is obtained by letting the observer ride with the mean local velocity, it seems quite clear that the Mach number effect at most would be an indirect one. The constants α_1, α_2, and α_3, being essentially correlation constants, ought to be nearly independent of the Mach number. The effect of Mach number on the mean temperature-velocity relationship is largely in the first term on the right-hand side of equation (19), where $M_1{}^2$ stands before the parenthesis. One may also expect the constants A_2 and Q to vary in some way with the Mach number through the frictional coefficient c_f and the heat-transfer coefficient c_q. However, the variations of c_f and c_q with Mach number are known to be rather slow. For an approximate theory, it might be sufficient to regard A_1, A_2, and Q also as universal constants independent of the free-stream Mach number. Likewise, the constants A_1 and A_2 might further be taken to be independent of the heat-transfer situation at the wall.

There are consequently three parameters α_1, α_2, and α_3 to be introduced in the problem besides the boundary conditions of the mean flow at the wall and in the free stream. These parameters are related to the turbulence mechanism and must be empirically determined by comparing with experimental data. Once determined from suitable experiments at low speeds with known heat transfer at the wall, they enable one to predict the behavior at any other Mach number or heat-transfer condition, provided, of course, that the turbulence level is still small and that the Mach number is not excessive.

In the following, equation (19) will be integrated first for the case of subsonic flow and then for the general case. For the subsonic case the solution can be expressed in an ascending series of M_1^2, and a few terms would be sufficient to compare with low-speed tests to deduce the values of the universal constants. The integral for the general case is useful mainly for the prediction of supersonic boundary layer after the universal constants are determined.

Case of subsonic flow.- Assume the solution to be expanded in ascending powers of M_1^2:

$$\theta = \theta_{(0)} = M_1^2\theta_{(1)} + M_1^4\theta_{(2)} + \ldots + M^{2n}\theta_{(n)} + \ldots \qquad (21)$$

hence,

$$\begin{aligned}\sqrt{\theta} = \theta_{(0)}^{1/2} &+ M_1^2\left[\frac{1}{2}\,\theta_{(0)}^{-1/2}\theta_{(1)}\right] + \\ &M_1^4\left[\frac{1}{2}\,\theta_{(0)}^{-1/2}\right]\left[\theta_{(2)} - \frac{1}{4}\,\theta_{(0)}^{-1}\theta_{(1)}^2\right] + \\ &M_1^6\left[\frac{1}{2}\,\theta_{(0)}^{-1/2}\right]\left[\theta_{(3)} - \frac{1}{2}\,\theta_{(0)}^{-1}\theta_{(1)}\theta_{(2)} + \right. \\ &\left.\frac{1}{8}\,\theta_{(0)}^{-2}\theta_{(1)}^3\right] + \ldots\end{aligned} \qquad (22)$$

Substituting into equation (19) and equating powers of M_1^{2n}, one obtains a set of differential equations for the functions $\theta_{(0)}$, $\theta_{(1)}$, . . ., $\theta_{(n)}$. Thus, for

$$\left.\begin{aligned}
M_1^0:&\qquad \frac{d\theta_{(0)}}{d\tilde{u}} = Q \\
M_1^2:&\qquad \frac{d\theta_{(1)}}{d\tilde{u}} = A_1\tilde{u} + A_2\theta_{(0)}^{1/2} \\
M_1^4:&\qquad \frac{d\theta_{(2)}}{d\tilde{u}} = \frac{1}{2}\,A_2\theta_{(0)}^{-1/2}\theta_{(1)} \\
M_1^6:&\qquad \frac{d\theta_{(3)}}{d\tilde{u}} = \frac{1}{2}\,A_2\theta_{(0)}^{-1/2}\left[\theta_{(2)} - \frac{1}{4}\,\theta_{(0)}^{-1}\theta_{(1)}^{-2}\right]
\end{aligned}\right\} \qquad (23)$$

and so forth. For boundary conditions one has for the incompressible case with no heat transfer $\theta = \theta_{(0)} = 1$ when $\tilde{u} = \tilde{u}_1$, where $\tilde{u}_1 = 1 - \left(1 - \frac{1}{\alpha_1}\right)\frac{\overline{u^*}}{\overline{u_1}}$. This result holds for any Mach number; therefore,

$$\theta_{(1)} = \theta_{(2)} = \ldots = 0 \quad \text{when} \quad \tilde{u} = \tilde{u}_1$$

With these conditions the integrals of equation (23) are found to be

$$\left.\begin{aligned}
\theta_{(0)} &= 1 + Q\left(\tilde{u} - \tilde{u}_1\right) \\
\theta_{(1)} &= \frac{1}{2} A_1\left(\tilde{u}^2 - \tilde{u}_1{}^2\right) + \frac{2A_2}{3Q}\left(\theta_{(0)}{}^{3/2} - 1\right) \\
\theta_{(2)} &= \frac{A_2}{2Q}\left\{\frac{A_1}{5Q^2}\left(\theta_{(0)}{}^{5/2} - 1\right) - \frac{2A_1(1 - Q)}{3Q^2}\left(\theta_{(0)}{}^{3/2} - 1\right) + \right. \\
&\quad \left.\left[\frac{A_1(1 - Q)^2}{Q^2} - A_1 - \frac{4A_2}{3Q}\right]\left(\theta_{(0)}{}^{1/2} - 1\right) + \frac{A_2}{3Q}\left(\theta_{(0)}{}^2 - 1\right)\right\}
\end{aligned}\right\} \quad (24)$$

and so forth. When the wall is insulated, $Q = 0$, and the functions $\theta_{(1)}$, $\theta_{(2)}$, . . . are simply polynominals of $\tilde{u}$. For convenient reference, a few of them are listed as follows:

$$\left.\begin{aligned}
\theta_{(0)} &= 1 \\
\theta_{(1)} &= \frac{A_1}{2}\left(\tilde{u}^2 - \tilde{u}_1{}^2\right) + A_2\left(\tilde{u} - \tilde{u}_1\right) \\
\theta_{(2)} &= \frac{A_2}{2}\left[\frac{A_1}{6}\left(\tilde{u}^3 - \tilde{u}_1{}^3\right) + \frac{A_2}{2}\left(\tilde{u}^2 - \tilde{u}_1{}^2\right) - \left(\frac{A_1}{2} + A_2\right)\left(\tilde{u} - \tilde{u}_1\right)\right] \\
\theta_{(3)} &= \frac{A_2}{2}\left\{-\frac{A_1{}^2}{80}\left(\tilde{u}^5 - \tilde{u}_1{}^5\right) + \left(\frac{A_1}{24} - \frac{A_1A_2}{16}\right)\left(\tilde{u}^4 - \tilde{u}_1{}^4\right) + \right. \\
&\quad \left[\frac{A_2}{6} - \frac{1}{12}\left(\frac{A_1{}^2}{2} + A_1A_2 - A_2{}^2\right)\right]\left(\tilde{u}^3 - \tilde{u}_1{}^3\right) - \\
&\quad \left[\frac{A_1}{4} + \frac{A_2}{2} - \frac{1}{3}\left(A_1A_2 + 2A_2{}^2\right)\right]\left(\tilde{u}_1 - \tilde{u}_1{}^2\right) + \\
&\quad \left.\left[\frac{A_1}{3} + \frac{A_2}{2} - \frac{1}{4}\left(\frac{A_1{}^2}{4} + A_1A_2 + A_2{}^2\right)\right]\left(\tilde{u} - \tilde{u}_1\right)\right\}
\end{aligned}\right\} \quad (25)$$

and so forth.

An interesting result to be noted is that the solution $\theta_{(0)}$ in equation (24) evidently coincides with what is usually known as Reynolds' analogy, where a similar transfer mechanism is assumed for both the momentum and the heat transfers. Here the same results are obtained because of the fact that the turbulent exchange, proportional to the temperature gradient $d\overline{T}/dy$, dominates the situation at very small Mach numbers. A similar transfer mechanism follows the assumed similar turbulence pattern. The solutions $\theta_{(1)}$, and so forth now give the correction to Reynolds' analogy at higher Mach numbers.

General case.- A brief outline will now be given for the general case. Rewrite equation (19):

$$\sqrt{\theta}\left(\frac{d\sqrt{\theta}}{d\tilde{u}} - \frac{A_2 M_1{}^2}{2}\right) = \frac{A_1 M_1{}^2}{2}\tilde{u} + \frac{Q}{2} \tag{26}$$

which is now homogeneous in $\sqrt{\theta}$ and $\tilde{u}$. Using the standard method, put

$$\sqrt{\theta} = F(\tilde{u})\left(\frac{A_1 M_1{}^2}{2}\tilde{u} + \frac{Q}{2}\right) \tag{27}$$

Then F is to be solved from

$$-\frac{F\,dF}{\frac{A_1 M_1{}^2}{2}F^2 - \frac{A_2 M_1{}^2}{2} - 1} = \frac{d\tilde{u}}{\frac{A_1 M_1{}^2}{2}\tilde{u} + \frac{Q}{2}} \tag{28}$$

Integrating equation (28), one gets

$$\frac{A_1 M_1{}^2}{2}\tilde{u} + \frac{Q}{2} = C\left(F - k_1\right)^{s_1}\left(F - k_2\right)^{s_2} \tag{29}$$

where

$$\left.\begin{aligned} k_1,k_2 &= \frac{1}{2}\left[\frac{A_2}{A_1} \pm \sqrt{\left(\frac{A_2}{A_1}\right)^2 + \frac{8}{A_1 M_1{}^2}}\right] \\ s_1 &= -k_1\Big/\left(k_1 - k_2\right) \\ s_2 &= k_2\Big/\left(k_1 - k_2\right) \\ C &= \text{Integration constant} \end{aligned}\right\} \tag{30}$$

Together, equations (27) and (29) form a parametric representation of the function $\theta = \theta(\tilde{u})$. Applying the condition at the outer edge, namely, $\theta = 1$ when $\tilde{u} = \tilde{u}_1$, one has

$$\left.\begin{aligned} F\left(\tilde{u}_1\right) &= \left(\frac{A_1 M_1{}^2}{2}\,\tilde{u}_1 + \frac{Q}{2}\right)^{-1} \\ &\text{and} \\ C &= \left(\frac{A_1 M_1{}^2}{2}\,\tilde{u}_1 + \frac{Q}{2}\right)\left[F\left(\tilde{u}_1\right) - k_1\right]^{s_1}\left[F\left(\tilde{u}_1\right) - k_2\right]^{-s_2} \end{aligned}\right\} \tag{31}$$

The complicated way in which the constants are entangled renders their evaluation from empirical data highly tedious. Besides, the proper form of equation (29), where every term is real, depends on the sign and magnitude of the constants and is best to be directly integrated from equation (28) once the constants are known.

VELOCITY AND TEMPERATURE DISTRIBUTIONS IN BOUNDARY LAYER

To determine the distribution of the mean velocity and/or the mean temperature across the boundary-layer thickness, it is necessary to bring in the definition of the length scale l_o as a function of y. In reference 2, two alternative expressions (equations (33) and (39)) have been given for the length scale, laying emphasis on the momentum and the energy equations, respectively. Integrating either will lead to a velocity distribution, from which a temperature distribution may be calculated by means of the results in the section "Velocity-Temperature Relation." The distributions evaluated from the two expressions will, in general, be different. It seems that only by comparison with experiments can one say whether both lead to essentially the same result or one form is preferable to the other in certain particular cases.

The two definitions are, nevertheless, of the same general form:

$$l_o \propto \frac{dX}{dy} \bigg/ \frac{d^2X}{dy^2} \tag{32}$$

with X standing for the variable $\overline{u}$ in equation (33) of reference 2 and for $\overline{T}$ in equation (39) of reference 2. Without specifying X, it follows from equation (32) of reference 2 that

$$\begin{aligned} v_o &\propto l_o \frac{d\overline{u}}{dy} \\ &= a_1 \frac{d\overline{u}}{dy} \frac{dX}{dy} \bigg/ \frac{d^2X}{dy^2} \end{aligned} \tag{33}$$

where a_1 is a proportional constant. Rewriting,

$$d\left(\log_e \frac{dX}{dy}\right) = a_1 \frac{d\overline{u}}{v_o}$$

Hence, by integration and substitution of equation (15),

$$\begin{aligned} \log_e \frac{dX}{dy} &= a_1 \int \frac{d\overline{u}}{v_o} \\ &= \frac{a_1}{\beta} \int \frac{d\overline{u}}{\sqrt{\overline{T}}} \end{aligned} \tag{34}$$

The temperature-velocity relation may then be used, yielding

$$\left.\begin{aligned} \frac{dX}{dy} &= G(\tilde{u}) \\ \text{where} \qquad G(\tilde{u}) &= \exp\left(\frac{a_1}{\beta}\int\frac{d\bar{u}}{\sqrt{\overline{T}}}\right) \end{aligned}\right\} \tag{35}$$

Finally, one may obtain the distribution by integrating equation (35)

$$\begin{aligned} y &= \int\frac{dX}{G(\tilde{u})} \\ &= \int\frac{dX}{d\theta}\frac{d\theta}{d\tilde{u}}\frac{d\tilde{u}}{G(\tilde{u})} \end{aligned} \tag{36}$$

For definition (33) of reference 2, $X = \tilde{u}$,

$$y = \int\frac{d\tilde{u}}{G(\tilde{u})} \tag{37}$$

for definition (39) of reference 2, $X = \theta$,

$$y = \int\frac{d\theta}{d\tilde{u}}\frac{d\tilde{u}}{G(\tilde{u})} \tag{38}$$

Again one may first solve the subsonic case for a more explicit expression, which must be reducible to the usual logarithmic distribution for incompressible flow. Then a brief discussion of the general case for any Mach number will be taken up.

Case of subsonic flow.- The function $G(\tilde{u})$ must first be evaluated before integrating equation (36). By definition (35),

$$\begin{aligned} G(\tilde{u}) &= \exp\left(\frac{a_1}{\beta}\int\frac{d\bar{u}}{\sqrt{\overline{T}}}\right) \\ &= \exp\left(a_1\sqrt{\frac{2}{c_f}}\int_{\tilde{u}_1}^{\tilde{u}}\frac{d\tilde{u}}{\sqrt{\theta}} + a_2\right) \end{aligned} \tag{39}$$

after nondimensionalizing with equation (8). Using equation (22), one has

$$
\left.
\begin{aligned}
&\int_{\tilde{u}_1}^{\tilde{u}} \frac{d\tilde{u}}{\sqrt{\theta}} = \varphi_0(\tilde{u}) + M_1^2\varphi_1(\tilde{u}) + M_1^4\varphi_2(\tilde{u}) + \dots \\
&\text{where} \\
&\varphi_0(\tilde{u}) = \int_{\tilde{u}_1}^{\tilde{u}} \theta_{(0)}^{-1/2}\, d\tilde{u} \\
&\varphi_1(\tilde{u}) = -\frac{1}{2}\int_{\tilde{u}_1}^{\tilde{u}} \theta_{(0)}^{-3/2}\,\theta_{(1)}\, d\tilde{u} \\
&\varphi_2(\tilde{u}) = \int_{\tilde{u}_1}^{\tilde{u}} \left(\frac{3}{8}\,\theta_{(1)}^{2}\theta_{(0)}^{-5/2} - \frac{1}{2}\,\theta_{(2)}\theta_{(0)}^{-3/2}\right) d\tilde{u} \\
&\varphi_3(\tilde{u}) = \int_{\tilde{u}_1}^{\tilde{u}} \left(\frac{3}{4}\,\theta_{(1)}\theta_{(2)}\theta_{(0)}^{-5/2} - \frac{1}{2}\,\theta_{(3)}\theta_{(0)}^{-3/2} - \right. \\
&\qquad\qquad \left. \frac{5}{16}\,\theta_{(1)}^{3}\theta_{(0)}^{-7/2}\right) d\tilde{u}
\end{aligned}
\right\} \qquad (40)
$$

and so forth, with the $\theta_{(n)}$'s defined by equation (24) with heat transfer at wall and by equation (25) with an insulated wall. Substituting into equation (39), one may therefore put

$$\left.\begin{aligned}
G(\tilde{u}) &= g_0(\tilde{u})\left[1 + M_1{}^2 g_1(\tilde{u}) + M_1{}^4 g_2(\tilde{u}) + \ldots\right] \\
&\text{where} \\
g_0(\tilde{u}) &= \exp\left(a_1\sqrt{\frac{2}{c_f}}\,\varphi_0(\tilde{u}) + a_2\right) \\
g_1(\tilde{u}) &= a_1\sqrt{\frac{2}{c_f}}\,\varphi_1(\tilde{u}) \\
g_2(\tilde{u}) &= a_1\sqrt{\frac{2}{c_f}}\left[\varphi_2(\tilde{u}) + \frac{a_1}{2}\sqrt{\frac{2}{c_f}}\,\varphi_1{}^2(\tilde{u})\right] \\
g_3(\tilde{u}) &= a_1\sqrt{\frac{2}{c_f}}\left[\varphi_3(\tilde{u}) + a_1\sqrt{\frac{2}{c_f}}\,\varphi_1(\tilde{u})\varphi_2(\tilde{u}) + \frac{a_1{}^2}{3c_f}\,\varphi_1{}^3(\tilde{u})\right] \\
&\quad \cdots\cdots\cdots
\end{aligned}\right\} \qquad (41)$$

Inverting,

$$\left.\begin{aligned}
\frac{1}{G(\tilde{u})} &= \frac{1}{g_0(\tilde{u})}\left[1 + M_1{}^2 g_{-1}(\tilde{u}) + M_1{}^4 g_{-2}(\tilde{u}) + \ldots\right] \\
&\text{where} \\
g_{-1}(\tilde{u}) &= -g_1(\tilde{u}) \\
g_{-2}(\tilde{u}) &= g_1{}^2(\tilde{u}) - g_2(\tilde{u})
\end{aligned}\right\} \qquad (42)$$

and so forth. Equation (42) may now be used to evaluate equations (37) and (38):

(1) Case (a) $X = \tilde{u}$:

$$\left.\begin{aligned}
y - y_r &= \int_{\tilde{u}_r}^{\tilde{u}} \frac{d\tilde{u}}{G(\tilde{u})} \\
&= Y_{u,0}(\tilde{u}) + M_1^2 Y_{u,1}(\tilde{u}) + M_1^4 Y_{u,2}(\tilde{u}) + \ldots
\end{aligned}\right.$$

where y_r is a reference station, $\tilde{u}_r$ is the value of $\tilde{u}$ at y_r, and

$$\left.\begin{aligned}
Y_{u,0} &= \int_{\tilde{u}_r}^{\tilde{u}} \frac{d\tilde{u}}{g_0(\tilde{u})} \\
Y_{u,1} &= -\int_{\tilde{u}_r}^{\tilde{u}} \frac{g_1(\tilde{u})}{g_0(\tilde{u})} d\tilde{u} \\
Y_{u,2} &= \int_{\tilde{u}_r}^{\tilde{u}} \frac{g_1^2(\tilde{u}) - g_2(\tilde{u})}{g_0(\tilde{u})} d\tilde{u} \\
&\ldots\ldots\ldots
\end{aligned}\right\} \tag{43}$$

It will be of interest to apply equation (43) to the incompressible case, namely $M_1^2 \longrightarrow 0$. The distribution is then

$$y - y_r = Y_{u,0}(\tilde{u})$$

For simplicity, one may take $y_r = \delta$, δ being the thickness of the boundary layer,[2] so that

$$\tilde{u}_r = \tilde{u}_1$$

$$\tilde{u} - \tilde{u}_1 = \frac{\bar{u}}{\bar{u}_1} - 1$$

With $\theta_{(0)}$ given by equation (24),

$$\varphi_0 = \int_{\tilde{u}_1}^{\tilde{u}} \frac{d\tilde{u}}{\sqrt{1 + Q\left(\tilde{u} - \tilde{u}_1\right)}} = \frac{2}{Q}\left[\sqrt{1 + Q\left(\tilde{u} - \tilde{u}_1\right)} - 1\right]$$

$$g_0 = \exp\left[a_1 \sqrt{\frac{2}{c_f}}\, \frac{2}{Q}\left(\sqrt{1 + Q\left(\tilde{u} - \tilde{u}_1\right)} - 1\right) + a_2\right]$$

Therefore

$$y - \delta = \int_0^X \exp\left[-\frac{2a_1}{Q}\sqrt{\frac{2}{c_f}}\left(\sqrt{1 + QX - 1}\right) - a_2\right] dX$$

$$= e^{-a_2}\left\{\left[-\frac{1}{a_1}\sqrt{\frac{2}{c_f}}\sqrt{1 + QX} - \frac{c_f}{4{a_1}^2}\right] \exp\left[-\frac{2a_1}{Q}\sqrt{\frac{2}{c_f}}\left(\sqrt{1 + QX - 1}\right)\right] - \left[-\frac{1}{a_1}\sqrt{\frac{c_f}{2}} - \frac{c_f}{4{a_1}^2}\right]\right\} \qquad (44)$$

---

[2]Actually this procedure does not give the best fit to experimental velocity distributions, since the similarity concept is not valid near the outer edge of the boundary layer. Instead of δ, the length scale should be taken as proportional to ν / u_τ, where u_τ is the frictional velocity. For general discussions, however, there ought to be little difference.

where

$$\chi = \tilde{u} - \tilde{u}_1 = \frac{\bar{u}}{\bar{u}_1} - 1$$

The constant a_2 may be determined from the usual condition at the wall:[3]

$$\frac{d\bar{u}}{dy} \longrightarrow \infty \quad \text{as} \quad y \longrightarrow 0$$

The other constant a_1 has to be evaluated by matching with experimental distribution. From the definition (33), it corresponds to Von Kármán's universal constant.

One may further develop the solution (44) into a power series in Q for cases where the heat transfer is relatively small. After some manipulation, the final result is

$$\left.\begin{aligned} \log_e\left(1 - \frac{y - \delta}{B_2}\right) &= -B_1\chi + Q\left(\frac{2 + B_1}{4}\chi\right) - \\ &\quad \left(\frac{\chi}{4\theta_1} + \frac{2 + B_1}{8}\chi^2\right)Q^2 + \ldots \\ \text{where} \qquad & \\ B_1 &= a_1\sqrt{\frac{2}{c_f}} \\ B_2 &= e^{-a_2}\left(1 + \frac{Q}{2B_1}\right)\Big/B_1 \end{aligned}\right\} \qquad (45)$$

[3]As a possible refinement, the laminar sublayer may be assumed to have a linear velocity profile, and, instead of infinity, the velocity gradient at the wall may be prescribed as

$$\frac{d\bar{u}}{dy} = \left(\frac{\tau_o}{\mu}\right)_{y=0} \quad \text{as} \quad y \longrightarrow 0$$

However, this step complicates the practical calculation and must be justified by experience.

Applying the condition of infinite velocity gradient at the wall, one finds easily

$$B_2 = -\delta \tag{46}$$

For the case of an insulated wall, there follows

$$\frac{\overline{u}}{\overline{u_1}} - 1 = -\frac{1}{B_1} \log_e \frac{y}{\delta} \tag{47}$$

which is precisely a form of the logarithmic law, matching with the experimental distribution at the assumed edge of the boundary layer. Such a form has been suggested by Dryden (reference 6) in 1935, where the frictional velocity u_τ and Von Kármán's constant K were used instead of a_1 and c_f in equation (47). By recalling the definition one verifies readily that

$$\left.\begin{aligned} u_\tau &= \sqrt{\frac{c_f}{2}} \\ a_1 &= K \end{aligned}\right\} \tag{48}$$

Equation (47) then reduces exactly to Dryden's form

$$\overline{u} - \overline{u_1} = -\frac{u_\tau}{K} \log_e \frac{y}{\delta} \tag{49}$$

One may also conclude that the logarithmic law, suitably modified, indeed can be approximately true even with the presence of a very small amount of heat transfer, since equation (45) shows that the first-order correction for $Q = 0$ merely amounts to replacing the factor in equation (49) by a slightly different one. If the heat transfer is appreciable, equation (45) predicts that the logarithmic law in the form of equation (49) would break down. There would be considerable interest if equation (45) could be tested for different values of Q.

It is apparent that each of the fractions $Y_{u,n}$ in equation (43) may be developed into a power series in Q as above. In general, therefore, the right-hand side of equation (46), which gives the velocity distribution, is in the form of a double series in the parameters Q and M_1^2, involving constants a_1 and a_2 which are to be determined once and for all. In fact, by comparing with the incompressible insulated case as above, both a_1 and a_2 are given in terms of B_1 and B_2.

For its practical importance, the results for the insulated case at subsonic Mach numbers are explicitly given as follows: Equations (40) are integrated by using equations (25),

$$\left.\begin{aligned}
\varphi_0(\tilde{u}) &= \tilde{u} - \tilde{u}_1 \\
\varphi_1(\tilde{u}) &= b_0 + b_1\tilde{u} + b_2\tilde{u}^2 + b_3\tilde{u}^3 \\
\varphi_2(\tilde{u}) &= c_0 + c_1\tilde{u} + c_2\tilde{u}^2 + c_3\tilde{u}^3 + c_4\tilde{u}^4 + c_5\tilde{u}^5
\end{aligned}\right\} \tag{50}$$

and so forth, where

$$\left.\begin{aligned}
b_0 &= -\frac{A_1}{6}\tilde{u}_1^{\,3} - \frac{A_2}{4}\tilde{u}_1^{\,2} \\
b_1 &= \frac{A_1}{4}\tilde{u}_1^{\,2} + \frac{A_2}{2}\tilde{u}_1 \\
b_2 &= -\frac{A_2}{4} \\
b_3 &= -\frac{A_1}{12} \\
c_0 &= -\frac{1}{20}A_1^{\,2}\tilde{u}_1^{\,5} - \frac{3}{16}A_1A_2\tilde{u}_1^{\,4} - \frac{5}{24}A_2^{\,2}\tilde{u}_1^{\,3} - \left(\frac{1}{16}A_1A_2 + \frac{1}{8}A_2^{\,2}\right)\tilde{u}_1^{\,2} \\
c_1 &= \frac{3}{32}A_1^{\,2}\tilde{u}_1^{\,4} + \frac{5}{12}A_1A_2\tilde{u}_1^{\,3} + \frac{1}{2}A_2^{\,2}\tilde{u}_1^{\,2} + \left(\frac{1}{8}A_1A_2 + \frac{1}{4}A_2^{\,2}\right)\tilde{u}_1 \\
c_2 &= -\frac{3}{16}A_1A_2\tilde{u}_1^{\,2} - \frac{3}{8}A_2^{\,2}\tilde{u}_1 - \frac{1}{16}A_1A_2 - \frac{1}{8}A_2^{\,2} \\
c_3 &= -\frac{1}{16}A_1^{\,2}\tilde{u}_1^{\,2} - \frac{1}{8}A_1A_2\tilde{u}_1 + \frac{1}{12}A_2^{\,2} \\
c_4 &= \frac{1}{12}A_1A_2 \\
c_5 &= \frac{3}{160}A_1^{\,2}
\end{aligned}\right\} \tag{51}$$

and so forth. The functions g_n of equations (41) become

$$\left.\begin{aligned}
g_0(\tilde{u}) &= \exp\left[B_1\left(\tilde{u} - \tilde{u}_1\right) + a_2\right] \\
g_1(\tilde{u}) &= B_1\varphi_1(\tilde{u}) \\
g_2(\tilde{u}) &= B_1\left[\varphi_2(\tilde{u}) + \frac{1}{2}B_1\varphi_1{}^2(\tilde{u})\right]
\end{aligned}\right\} \tag{52}$$

and so forth. Hence,

$$\left.\begin{aligned}
Y_{u,0} &= B_2\left\{1 - \exp\left[-B_1\left(\tilde{u} - \tilde{u}_1\right)\right]\right\} \\
Y_{u,1} &= B_2\left\{\exp\left[-B_1\left(\tilde{u} - \tilde{u}_1\right)\right]\left(d_{10} + d_{11}\tilde{u} + d_{12}\tilde{u}^2 + d_{13}\tilde{u}^3\right) - \right. \\
&\qquad \left.\left(d_{10} + d_{11}\tilde{u}_1 + d_{12}\tilde{u}_1{}^2 + d_{13}\tilde{u}_1{}^3\right)\right\} \\
Y_{u,2} &= B_2\left\{-\exp\left[-B_1\left(\tilde{u} - \tilde{u}_1\right) - a_2\right]\left(d_{20} + d_{21}\tilde{u} + d_{22}\tilde{u}^2 + \right.\right. \\
&\qquad \left. d_{23}\tilde{u}^3 + d_{24}\tilde{u}^4 + d_{25}\tilde{u}^5 + d_{26}\tilde{u}^6\right) + \left(d_{20} + d_{21}\tilde{u}_1 + \right. \\
&\qquad \left.\left. d_{22}\tilde{u}_1{}^2 + d_{23}\tilde{u}_1{}^3 + d_{24}\tilde{u}_1{}^4 + d_{25}\tilde{u}_1{}^5 + d_{26}\tilde{u}_1{}^6\right)\right\}
\end{aligned}\right\} \tag{53}$$

and so forth, where

$$\left.\begin{aligned}
d_{10} &= b_0 + \frac{b_1}{B_1} + \frac{2b_2}{B_1{}^2} + \frac{6b_3}{B_1{}^3} \\
d_{11} &= b_1 + \frac{2b_2}{B_1} + \frac{6b_3}{B_1{}^2} \\
d_{12} &= b_2 + \frac{3b_3}{B_1} \\
d_{13} &= b_3
\end{aligned}\right\} \tag{54}$$

and

$$d_{2,n} = \sum_{m=0}^{6-n} \frac{(m + n)!}{n!} \frac{h_m}{B_1^{m-1}} \tag{55}$$

The constants h_m are defined as follows:

$$\left.\begin{aligned}
h_0 &= -c_0 + \frac{B_1}{2} b_0^2 \\
h_1 &= -c_1 + B_1 b_0 b_1 \\
h_2 &= -c_2 + \frac{B_1}{2}\left(b_1^2 + 2b_0 b_2\right) \\
h_3 &= -c_3 + B_1\left(b_1 b_2 + b_0 b_3\right) \\
h_4 &= -c_4 + \frac{B_1}{2}\left(b_2^2 + 2b_1 b_3\right) \\
h_5 &= -c_5 + B_1 b_2 b_3 \\
h_6 &= \frac{B_1}{2} b_3^2
\end{aligned}\right\} \tag{56}$$

With these evaluations, the distribution for the insulated subsonic case becomes finally,

$$\left.\begin{aligned}
&\log_e \frac{y}{\delta} = -B_1\left(\frac{\overline{u}}{\overline{u}_1} - 1\right) + M_1^2 f_1(\tilde{u}) + M_1^4 f_2(\tilde{u}) + \ldots \\
&\text{where} \\
&f_1(\tilde{u}) = -\sum_{n=0}^{3} d_{1n}\left(\tilde{u}^n - \tilde{u}_1^n\right) \\
&f_2(\tilde{u}) = \sum_{n=0}^{6}\left(d_{2n} - \frac{1}{2}\sum_{l+m=n} d_{1l} d_{1m}\right)\left(\tilde{u}^n - \tilde{u}_1^n\right)
\end{aligned}\right\} \tag{57}$$

and so forth where B_2 is now determined again by the condition of infinite velocity gradient at wall,

$$\frac{\delta}{B_2} = -1 + M_1^2 \sum_{n=0}^{3} d_{1n}\tilde{u}_1^n - M_1^4 \sum_{n=0}^{6} d_{2n}\tilde{u}_1^n + \ldots \tag{58}$$

The other alternative for X will now be considered:

(2) Case (b) $X = \theta$:

Instead of equation (43), the equation becomes

$$y - y_r = \int_{\tilde{u}_r}^{\tilde{u}} \frac{d\theta}{d\tilde{u}} \frac{d\tilde{u}}{G(\tilde{u})} \tag{59}$$

Since, from equation (43),

$$\frac{1}{G(\tilde{u})} = \frac{d}{d\tilde{u}} \sum_{n=0}^{\infty} Y_{u,n} M_1^{2n}$$

$$= \sum_{n=0}^{\infty} M_1^{2n} \frac{d}{d\tilde{u}} Y_{u,n}$$

one may rewrite equation (59) as

$$y - y_r = Y_{\theta,0}(\tilde{u}) + M_1^2 Y_{\theta,1}(\tilde{u}) + M_1^4 Y_{\theta,2}(\tilde{u}) + \ldots$$

where

$$Y_{\theta,n} = \int_{\tilde{u}_r}^{\tilde{u}} \sum_{l+m=n} \frac{d\theta_{(l)}}{d\tilde{u}} \frac{dY_{u,m}}{d\tilde{u}} d\tilde{u} \tag{60}$$

In parallel with case (a), where $X = \tilde{u}$, the results for incompressible flow with heat transfer and for an insulated wall in subsonic flow are given in the following. For the former problem, by using equation (23) one obtains immediately

$$Y_{\theta,0} = QY_{u,0} \tag{61}$$

so that a modification of the definition of B_2 leads to the same distribution (45). Thus the two choices of X are equivalent to each other even for the distribution in incompressible flow.

For the compressible case with an insulated wall, the right-hand side of equation (60) starts with $Y_{\theta,1}$. Now

$$Y_{\theta,1} = \int_{\tilde{u}_1}^{\tilde{u}} \frac{d\theta_{(1)}}{d\tilde{u}} \frac{dY_{u,0}}{d\tilde{u}} \, d\tilde{u}$$

$$= \int_{\tilde{u}_1}^{\tilde{u}} \left(A_1\tilde{u} + A_2\right) \frac{dY_{u,0}}{d\tilde{u}} \, d\tilde{u}$$

by taking $y_r = \delta$ and using equations (23) and (25). Integrating,

$$Y_{\theta,1} = -B_2\left[\left(A_2 + \frac{A_1}{B_1} + A_1\tilde{u}\right)e^{-B_1\left(\tilde{u}-\tilde{u}_1\right)} - \left(A_2 + \frac{A_1}{B_1} + A_1\tilde{u}_1\right)\right]$$

Substituting into expression (59), one has

$$y - \delta = M_1{}^2Y_{\theta,1} + O\left(M_1{}^4\right)$$

The condition of infinite velocity gradient at $y = 0$ again determines B_2,

$$\delta = -B_2M_1{}^2\left(A_2 + \frac{A_1}{B_1} + A_1\tilde{u}_1\right) + O\left(M_1{}^4\right)$$

Hence the distribution may be written as

$$\log_e \frac{y}{\delta} = -B_1\left(\frac{\bar{u}}{\bar{u}_1} - 1\right) + \log_e\left[1 + \frac{A_1\left(\frac{\bar{u}}{\bar{u}_1} - 1\right)}{A_2 + \frac{A_1}{B_1} + A_1\tilde{u}_1}\right] + O\left(M_1{}^2\right) \tag{62}$$

Equation (62) indicates that in the absence of heat transfer at the wall, the choice of $X = \theta$ leads to an additional term to the usual logarithmic law even when $M_1^2 \ll 1$ (but not zero). The additional term is of more weight nearer the wall where $\bar{u}/\bar{u}_1$ differs appreciably from unity. Noticing further that A_1/A_2 is a controlling parameter,

$$\frac{A_1}{A_2} = \frac{\alpha_1/\alpha_3}{\sqrt{c_f/2}}$$

by equation (20), one may say that the predicted deviation from the usual logarithmic law depends on the relation between the heat generated by dissipation and the turbulent transfer of such heat (and due to compressibility). Ordinary theories assume the dissipation to be negligible, equivalent to putting $A_1 = 0$. Then the additional term vanishes.

The apparent contradiction of equations (45) and (62) when only the lowest-order terms in both are retained is resolved by the observation that, in complete form, one should have

$$y - \delta = QY_{u,0} + M_1^2 Y_{\theta,1} + \ldots \tag{63}$$

so that the lowest-order terms should be taken according to whether $Q \ll M_1^2$ or the converse is true. If both are of the same order, neither should be left out and the final expression certainly will contain an additional term analogous to that in equation (62). Physically, it is obvious that if a great amount of heat conduction is present at the wall, it certainly would overshadow the distributed heating due to dissipation (and compressibility) at lower Mach numbers. On the other hand, when there is no external heating, the dissipation becomes the only predominant factor.

The Mach number effect in equation (62) can be worked out without difficulty, but, involving no new features, the details are omitted in this report.

General case.- The general case will be only briefly treated, as the integration depends on the values of the constants A_1, A_2, and Q.

(Cf. equations (28) to (30).) Still starting with equations (36) and (39), one substitutes $\sqrt{\theta}$ by equation (27)

$$G(\tilde{u}) = \exp\left[B_1 \int_{\tilde{u}_1}^{\tilde{u}} \frac{d\tilde{u}}{F(\tilde{u})\left(\frac{A_1 M_1{}^2}{2}\tilde{u} + \frac{Q}{2}\right)} + a_2\right]$$

$$= \exp\left(-B_1 \int_{F_1}^{F} \frac{dF}{\frac{A_1 M_1{}^2}{2}F^2 - \frac{A_2 M_1{}^2}{2}F - 1} + a_2\right)$$

$$= B_3\left(\frac{F - k_1}{F - k_2}\right)^{\beta_1}$$

where

$$\left.\begin{aligned} \beta_1 &= \frac{2}{A_1 M_1{}^2}\,\frac{1}{k_2 - k_1} \\ B_3 &= e^{a_2}\left[\frac{F(\tilde{u}_1) - k_2}{F(\tilde{u}_1) - k_1}\right]^{\beta_1} \end{aligned}\right\} \qquad (64)$$

Hence, for $X = \tilde{u}$,

$$y - y_r = \int_{\tilde{u}_r}^{\tilde{u}} \frac{d\tilde{u}}{G(\tilde{u})}$$

$$= \frac{1}{B_3}\int_{F_r}^{F} \frac{d\tilde{u}}{dF}\left(\frac{F - k_1}{F - k_2}\right)^{-\beta_1} dF \qquad (65)$$

with $d\tilde{u}/dF$ given by equations (28) and (29). Similarly, for $X = \theta$,

$$y - y_r = \frac{1}{B_3} \int_{F_r}^{F} \frac{d\theta}{d\tilde{u}} \frac{d\tilde{u}}{dF} \left(\frac{F - k_1}{F - k_2} \right)^{-\beta_1} dF \tag{66}$$

with $d\theta/d\tilde{u}$ to be evaluated from equations (27) and (29). The constant of integration B_3 is determined from the condition:

$$\frac{d\overline{u}}{dy} \longrightarrow \infty \quad \text{as} \quad y \longrightarrow 0$$

The constant B_1 has previously been identified with u_τ/K.

CORRELATION OF THEORY WITH EXPERIMENTS AND DISCUSSIONS

Determination of Arbitrary Constants

There are two sets of arbitrary constants. In the temperature-velocity relation, one has three constants α_1, α_2, and α_3 representing essentially combinations of the correlations, which are assumed universal with regard to the free-stream Mach number and the heat-conduction conditions at wall. In addition there is the mean velocity $\overline{u^*}$ at the edge of the laminar sublayer separating the turbulent boundary layer from the wall. The second set arises out of the integration for the mean velocity distribution within the boundary layer, consisting of B_1 and B_2 in the section "Velocity and Temperature Distributions in Boundary Layer." The external conditions defining the mean flow are the following: The free-stream velocity, pressure, temperature, the friction at wall, and the temperature at wall (or the amount of heat conduction from the wall).

With given external conditions, the constants can be determined by comparing with accurate experimental results. The significant point is that, once determined, a universal law for all Mach numbers and heat conduction is established. No other ad hoc assumptions need to be introduced for special cases.

It is then only necessary to conduct a few low-speed experiments with prescribed heat conduction at the wall and measure the mean temperature and velocities at points across the section. Éliás' data (reference 7) were in this category but unfortunately were not presented in enough detail to suit the present purpose. One may naturally also deduce from high-speed measurements with somewhat more work. Wilson (reference 8) and Ladenburg and Bershader (reference 9) published the results for a supersonic free stream. Unfortunately, again, their data were not sufficient, because only one of the two variables $\overline{u}$ and $\overline{T}$ was measured in either case. Both assumed the isoenergetic relation to hold for evaluation of the other variable.

There are arguments for the validity of the isoenergetic relation, either from the practical reason that the effective Prandtl number is nearly unity or from some assumption on the mixing lengths for momentum and energy, such as Ferrari's (reference 5). Even admitting the arguments, no isoenergetic law would follow when the dissipation term is kept in the energy equation. In reference 2, it has been shown that the dissipation terms should be retained for a consistent theory, and one could only regard the success, if any, to be essentially an empirical one for the Mach number or heat conduction involved. Such empirical results might be useful, however, for the approximate evaluation of the constants α_1, α_2, and α_3, at least as a guidance to the orders of magnitude.

The constant $\overline{u^*}$ is a more troublesome one. In appendix B, it is shown to be dependent on the Mach number as well as another empirical constant defining the extent of the turbulent layer. The correct determination, therefore, is very tedious and perhaps not warranted because of the approximate nature of the theory. A possible way is to determine its value from incompressible flow and neglect the dependence on Mach number. The error introduced can be assessed only by comparing with experimental results.

The determination of B_1 and B_2 is done by matching with the empirical distribution of the mean velocity or temperature. In the section "Velocity and Temperature Distributions in Boundary Layer" it is found that B_2 is essentially determined by the condition that $d\overline{u}/dy \longrightarrow \infty$ as $y \longrightarrow 0$, and B_1 corresponds to u_τ/K in the usual incompressible case (cf. equation (47) and (48)), K being Von Kármán's universal constant. Approximately, within the range of moderate Mach numbers, B_1 might be taken to be independent of Mach number without serious error.

Some Experimental Results of Turbulent

Boundary Layer

A brief examination of the velocity distribution in the incompressible case may be made to check with the logarithmic law predicted by similarity theory. A quite common form for turbulent-boundary-layer flow has been the 1/7-power law. Recently, there are the careful measurements in the National Bureau of Standards[4] and some others such as those by Hama (reference 10). Typical examples from these works at various Reynolds numbers have been plotted on semilogarithmic paper as figure 1. It is found that outside the "viscous" layer near the wall, all the curves tend to show a more or less straight portion, of approximately the same slope, the extent of the straight portion decreasing as the Reynolds number decreases.[5] Such a phenomenon is in accordance with the expectation on the basis of the similarity theory, because a smaller Reynolds number brings in the viscous effects to a greater degree, causing a merging of the higher- and lower-frequency parts of the energy spectrum. The slope was predicted to be proportional to u_τ and hence should change only slowly with Reynolds number, again verified by the experiments. For larger values of y/δ, the velocity curve deviates from the straight portion on the semilogarithmic plot and may be approximated by a power law. In this part of the boundary layer, the turbulence is nearly isotropic and therefore the mean flow is not expected to follow closely the logarithmic relation. The empirical 1/7-power law is also included and shows good agreement on the whole.

For the compressible case, the works of Wilson (reference 8) and Ladenburg and Bershader (reference 9) have been quoted before. The velocity distributions given are also plotted in figure 1 for comparison. It should be noted that in deducing the velocity distribution from their measurements, both made use of the isoenergetic relation. Wilson's measurements were done with a pitot tube and registered actually the Mach number variation within the boundary layer, computed from the stagnation pressures. A discussion was made on the error introduced by the assumed isoenergetic relation, with the conclusion that the maximum error amounts to about 3 percent for $M_1 = 2.0$. Now Ladenburg and Bershader measured by means of an interferometer and, therefore,

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[4]As yet unpublished. The authors are obliged to Dr. Schubauer, chief of the Aerodynamics Section in NBS, for furnishing the data.

[5]Hama's results show more waviness than those of NBS, partially because his Reynolds number is quite low, and are omitted in figure 1.

have recorded actually the temperature variation. By coincidence the two sets of experiments have some of the tests made under almost identical conditions, like the following:

Reference 9: $T_1 = 143^\circ$ C, $M_1 = 2.3$, $R_\delta \approx 10^5$

Reference 8:[6] $T_1 = 150^\circ$ C, $M_1 = 2.03$, $R_\delta \approx 10^5$

After correcting for the Mach number difference, one may estimate the temperature distribution in Wilson's case from Ladenburg and Bershader's data and evaluate the velocity from the measured local Mach number. In correcting for the Mach number difference, assume a modified isoenergetic law to hold,

$$\frac{\overline{T}}{\overline{T}_1} = 1 + aM_1^2\left(1 - \frac{\overline{u}^2}{\overline{u}_1^2}\right)$$

a being a constant, so that

$$\Delta\left(\frac{\overline{T}}{\overline{T}_1}\right) = \left(\frac{\overline{T}}{\overline{T}_1} - 1\right)\Delta M_1^2/M_1^2$$

Then, letting subscripts refer to the respective experiments,

$$\left(\frac{\overline{T}}{\overline{T}_1}\right)_W \approx \left(\frac{\overline{T}}{\overline{T}_1}\right)_{L\text{ and }B} + \left[\left(\frac{\Delta\overline{T}}{\overline{T}_1}\right)_{L\text{ and }B} - 1\right]\frac{\left(M_1^2\right)_W - \left(M_1^2\right)_{L\text{ and }B}}{\left(M_1^2\right)_{L\text{ and }B}}$$

and, finally,

$$\left(\frac{\overline{u}}{\overline{u}_1}\right)_W \approx \left(\frac{M}{M_1}\right)_W\sqrt{\left(\frac{\overline{T}_1}{\overline{T}}\right)_W}$$

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[6]Table I, station 6, of reference 8.

The following results are then obtained:

y/δ	0.104	0.217	0.321	0.477	0.686
$\left(M/M_1\right)_W$	.665	.751	.820	.903	.982
$\left(\overline{T}/\overline{T_1}\right)_W$	1.31	1.24	1.20	1.12	1.05
$\left(\overline{u}/\overline{u}_1\right)_W$	.762	.836	.899	.958	1.003
$\left(\overline{u}/\overline{u}_1\right)_{WW}$	.769	.838	.888	.943	.990

where the subscripts WW refer to the value given by Wilson using the isoenergetic law. It is interesting to see that, at least for this particular case, the isoenergetic law does give a very close approximation as checked by the independent temperature measurement. However, whether the same agreement would result in other cases cannot be readily ascertained.

Variation of Skin Friction with Reynolds and Mach Numbers

In the present theory the skin friction was introduced as an external condition such as would influence the turbulence pattern through its effects on the mean distributions. One may stretch the theory a little to yield an equation of the skin friction involving the chordwise Reynolds number and the Mach number, as Prandtl and Von Kármán had done for the incompressible case (references 11 and 12). A momentum thickness ϑ for the entire boundary layer may be derived, which is related to the skin friction by

$$\tau = \overline{\rho_1}\, \overline{u_1}^2 \frac{d\vartheta}{dx}$$

The viscosity is introduced to form a length scale with the frictional velocity u_τ and a chordwise Reynolds number appears. When the universal constants in the present theory are known, the skin-friction relation thus derived will also give explicit Mach number effects. A point to be noted is that the velocity distribution should refer to the frictional velocity u_τ, instead of $\overline{u_1}$ at the outer edge $y = \delta$. For simplicity it might be necessary to make approximations but there seems to be no great difficulty.

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APPENDIX A

SYMBOLS

a_1, a_2	constants, defined by equations (33) and (39), respectively
$b_0, b_1, \ldots$ $c_0, c_1, \ldots$	coefficients in expansions for φ_1 and φ_2 defined in equations (51)
c_f, c_q	coefficients of skin friction and heat transfer, respectively, defined in equations (18)
c_p, c_v	specific heats at constant pressure and constant volume, respectively
$d_{10}, d_{11}, \ldots$ $d_{20}, d_{21}, \ldots$	coefficients in expansions for $Y_{u,1}$ and $Y_{u,2}$, defined by equations (54) and (55), respectively
e	dilatation $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$
f_1, f_2	correction functions for Mach number effect on logarithmic law, defined by equations (57)
$g_0, g_1, \ldots$	functions in expansion for G, defined by equations (41)
$g_0, g_{-1}, \ldots$	functions in expansion for $1/G$, defined by equations (42)
$h_0, h_1, \ldots$	constants defined by equations (56)
k	coefficient of heat conductivity of fluid
k_1, k_2	constants defined by equations (30)
l_0	similarity scale of length
m	rate of mass transfer at wall

$\overline{p}, \overline{p_1}$	mean pressure and mean pressure in free stream, respectively
q	rate of heat transfer at wall
q^*	rate of amount of heat generated in laminar sublayer
s_1, s_2	constants defined by equations (30)
t	time
t_o	similarity scale of time for fluctuations
u, v, w	velocity components in x-, y-, and z-directions, respectively
v_o	similarity scale of velocity fluctuations
$\overline{u_e}$	modified mean velocity, defined by equation (16)
$\overline{u^*}$	magnitude of mean velocity at edge of laminar sublayer
$\tilde{u}$	nondimensionalized modified mean velocity $\left(\tilde{u} = \overline{u_e}/\overline{u_1}\right)$
$\tilde{u}_1$	value of $\tilde{u}$ evaluated at outer edge of turbulent boundary layer
$\tilde{u}_r$	value of $\tilde{u}$ evaluated at a reference station within boundary layer
u_τ	frictional velocity $\left(\sqrt{\tau_o/\overline{\rho_1}}\right)$
x, y, z	Cartesian coordinates; x, axis in direction of plate and free stream; y, axis normal to plate; and z, axis parallel to leading edge of plate
y_r	reference station in boundary layer
A_1, A_2	constants defined by equations (20)
B_1, B_2	constants defined by equations (45)
C	constant defined by equations (31)
F	function defined by equation (27)
G	function defined by equation (35)
M_1	Mach number of free stream

Q	constant defined by equations (20)
R	gas constant in equation of state
R_{δ}	Reynolds number based on boundary-layer thickness
T	temperature
$\overline{T_o}$	mean temperature at wall
$\overline{U}$	mean velocity component in x-direction
X	function defining similarity scale of length, as in equation (32)
$Y_{u,0}, Y_{u,1}, \ldots$	distribution functions for mean velocity, defined by equations (43)
$Y_{\theta,0}, Y_{\theta,1}, \ldots$	distribution functions for mean velocity, defined by equations (60)
$\alpha_1, \alpha_2, \alpha_3$	constants representing correlations in averaged equation of energy
β	constant $\left(\sqrt{R \frac{\tau_o}{\overline{p}}}\right)$
β_1	constant defined by equations (64)
γ	ratio of specific heats
δ	thickness of boundary layer
δ^*	thickness of laminar sublayer
ϵ	rate of dissipation
θ	nondimensionalized mean temperature, defined by equations (18)
θ_1, θ_2	similarity scales of temperature fluctuation
$\theta_{(0)}, \theta_{(1)}, \theta_{(2)}, \ldots$	functions in expansion for θ, defined by equations (24) or (25)

ϑ	momentum thickness of boundary layer
μ	coefficient of viscosity
ν	coefficient of kinematic viscosity
ρ	density of fluid
τ	shearing stress
τ_0, τ_{0z}	shearing stress at the wall in direction of main stream and parallel to leading edge, respectively
$\varphi_0, \varphi_1, \ldots$	functions in expansion for F, defined by equations (40)

The subscript 1 denotes quantities in the free stream. Barred quantities always represent mean values; primed quantities represent fluctuations.

APPENDIX B

DISSIPATION IN THE VISCOUS SUBLAYER

Imagine the boundary layer to be idealized to consist of two parts: The major part is entirely turbulent, extending from the outer edge to a certain distance close to the wall. From there on the viscous effects take over completely, and a viscous sublayer is formed. In the viscous sublayer, the flow is laminar and free of turbulence.

Let the thickness of the viscous sublayer be δ^*. The energy equation within the sublayer then degenerates into

$$\mu\left(\frac{d\overline{u}}{dy}\right)^2 + k\,\frac{d^2\overline{T}}{dy^2} = 0 \tag{B1}$$

since all the correction terms vanish for the laminar flow with $\overline{u} = \overline{u}(y)$, $\overline{v} = 0$. The first term is the viscous dissipation; the second is the heat conduction. Noting that $\mu\,\frac{d\overline{u}}{dy} = \tau_0 = \text{Constant}$, one gets by integration between 0 and δ^*,

$$\tau_0\overline{u^*} - k\,\frac{d\overline{T}}{dy}\bigg|_{y=0} = q_1 \tag{B2}$$

assuming conduction to be comparatively negligible at $y = \delta^*$. Obviously $\tau_0\overline{u^*}$ is the heat generated in the viscous sublayer by dissipation, and $-k\,\frac{d\overline{T}}{dy}\Big|_{y=0}$ is the heat conducted into the fluid from the wall. In the notation of equation (9), they are the quantities q^* and q, respectively. Therefore, q_1 is the resultant heat exchange between the viscous and the turbulent layers.

It is important to estimate, at least approximately, the magnitude $\overline{u^*}$ for the present theory (cf. equation (16)). To do so the first step is to estimate the thickness δ^*. Usually the thickness of the sublayer is expressed by

$$\frac{\delta^* u_\tau}{\nu} = h, \quad \text{a constant} \tag{B3}$$

In the actual case there is a transition layer where the viscous and turbulent shears are of the same order of magnitude. The value of h in the incompressible case varies from approximately 10 to 30, depending on whether the sublayer is taken to be strictly laminar and has, consequently, a linear velocity profile, or the sublayer is to consist of the entire thickness where noticeable departure from the turbulent velocity profile occurs. In the idealization adopted in this section, the transition layer is omitted and the boundary of the "viscous layer" should therefore lie somewhere within the actual transition layer. By such reasoning, it seems that an average value of h of the order of 20 is probably adequate for the purpose.

It may be seen that, with such a concept, the value of h ought to vary with the Mach number. For, one may regard the sublayer as owing its existence to the fact that the viscous shear becomes an appreciable part of the total shear. Suppose, then, that h is defined to satisfy

$$\mu \left.\frac{d\bar{u}}{dy}\right|_{y=\delta^*} = \tau_o \alpha \tag{B4}$$

α being a constant less than unity. The quantity $d\bar{u}/dy$ may be estimated by using the turbulent velocity distribution. For instance, in the subsonic case if one uses equation (57),

$$\bar{u} = \frac{u_\tau}{K} \log_e y + B + M_1{}^2 f_1(\tilde{u}) + M_1{}^4 f_2(\tilde{u}) + \ldots$$

$$\frac{d\bar{u}}{dy} = \frac{u_\tau}{Ky}\left[1 + M_1{}^2 f_1{}'(\tilde{u}) + \ldots\right]$$

Hence, by substituting into equation (B4),

$$h = \frac{1}{\alpha K}\left(1 + M_1{}^2 f_1{}'(\tilde{u}^*) + \ldots\right)$$

where $\tilde{u}^* = \frac{1}{\alpha_1}\overline{u^*}$ (cf. equation (16)). If h_i is the value for the incompressible case, there follows

$$\frac{h}{h_i} = 1 + M_1^2 f_1'(\tilde{u}^*) + \ldots \tag{B5}$$

Since the right-hand side involves $\overline{u^*}$, successive approximations might be necessary for obtaining the solution.

Having determined δ^*, one may find $\overline{u^*}$ by again applying the turbulent distribution.

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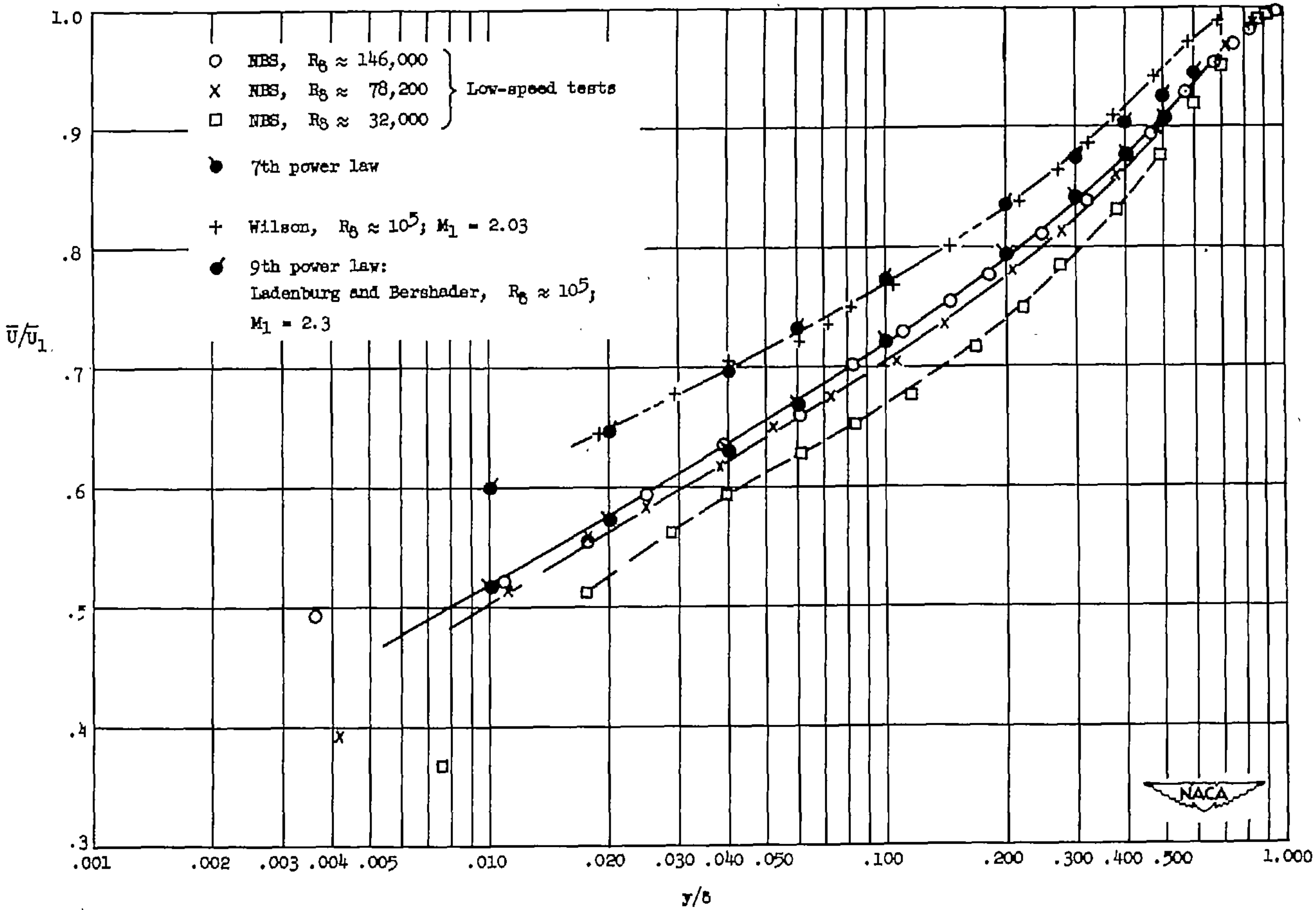

Figure 1.- Experimental velocity distributions.

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